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THE GENERALIZED METHOD OF QUASILINEARIZATION AND NONLINEAR BOUNDARY VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. The generalized method of quasilinearization is applied to obtain a monotone sequence of iterates converging uniformly and rapidly to a solution of second order nonlinear boundary value problem with nonlinear integral boundary conditions.

1. Introduction

In this paper, we shall study the method of quasilinearization for the nonlinear boundary value problem with integral boundary conditions

$$
\begin{aligned}
&x''(t) = f(t,x),\ t \in J = [0,1],\\
&x(0) - k_1 x'(0) = \int_0^1 h_1(x(s))ds, \qquad (1.1)\\
&x(1) + k_2 x'(1) = \int_0^1 h_2(x(s))ds,
\end{aligned}
$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ and $h_i : \mathbb{R} \to \mathbb{R}\,(i = 1, 2)$ are continuous functions and k_i are nonnegative constants. Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint and nonlocal boundary value problems as special cases. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers [13, 14, 15] and the references therein. Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors, for example [11, 12, 16, 17].

The purpose of this paper is to develop the method of quasilinearization for the boundary value problem (1.1). The main idea of the method of quasilinearization as developed by Bellman and Kalaba [1] and generalized by Lakshmikantham [4, 5] has been applied to a variety of problems [3, 6, 7]. Recently, Eloe and Gao [8], Ahmad, Khan and Eloe [9] have developed the quasilinearization method for three point boundary value problems. More recently, Khan, Ahmad [10] developed the method to treat first order problems

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with integral boundary conditions

$$x'(t) = f(t, x(t)),\ t \in [0, T]$$
$$x(0) = ax(T) + \int_0^T b(s)x(s)ds + k = Bx + k.$$

In the present paper we extend the method of generalized quasilinearization to the boundary value problem (1.1) and we obtain a sequence of solutions converging uniformly and rapidly to a solution of the problem.

2. Preliminaries

We know that the homogeneous problem

$$x''(t) = 0,\ \ t \in J = [0, 1],$$
$$x(0) - k_1 x'(0) = 0, \quad x(1) + k_2 x'(1) = 0,$$

has only the trivial solution. Consequently, for any $\sigma(t)$, $\rho_1(t)$, $\rho_2(t) \in C[0, 1]$, the corresponding nonhomogeneous linear problem

$$x''(t) = \sigma(t),\ \ t \in J = [0, 1],$$
$$x(0) - k_1 x'(0) = \int_0^1 \rho_1(s)ds, \quad x(1) + k_2 x'(1) = \int_0^1 \rho_2(s)ds,$$

has a unique solution $x \in C^2[0, 1]$,

$$x(t) = P(t) + \int_0^1 G(t, s)\sigma(s)ds,$$

where

$$P(t) = \frac{1}{1 + k_1 + k_2}\{(1 - t + k_2)\int_0^1 \rho_1(s)ds + (k_1 + t)\int_0^1 \rho_2(s)ds\}$$

is the unique solution of the problem

$$x''(t) = 0,\ \ t \in J = [0, 1],$$
$$x(0) - k_1 x'(0) = \int_0^1 \rho_1(s)ds,$$
$$x(1) + k_2 x'(1) = \int_0^1 \rho_2(s)ds,$$

and

$$G(t, s) = \frac{-1}{k_1 + k_2 + 1}\begin{cases} (k_1 + t)(1 - s + k_2), & 0 \le t < s \le 1 \\ (k_1 + s)(1 - t + k_2), & 0 \le s < t \le 1 \end{cases}$$

is the Green's function of the problem. We note that $G(t, s) < 0$ on $(0, 1) \times (0, 1)$.

Definition 2.1. Let $\alpha, \beta \in C^2[0,1]$. We say that α is a lower solution of (1.1) if

$$\begin{aligned}
&\alpha'' \geq f(t, \alpha(t)), \quad t \in [0,1] \\
&\alpha(0) - k_1\alpha'(0) \leq \int_0^1 h_1(\alpha(s))ds, \\
&\alpha(1) + k_2\alpha'(1) \leq \int_0^1 h_2(\alpha(s))ds.
\end{aligned}$$

Similarly, β is an upper solution of the BVP (1.1), if β satisfies similar inequalities in the reverse direction.

Now, we state and prove the existence and uniqueness of solutions in an ordered interval generated by the lower and upper solutions of the boundary value problem.

Theorem 2.2. *Assume that α and β are respectively lower and upper solutions of* (1.1) *such that $\alpha(t) \leq \beta(t)$, $t \in [0,1]$. If $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $h_i : \mathbb{R} \to \mathbb{R}\,(i = 1,2)$ are continuous and $h_i'(x) \geq 0$, then there exists a solution $x(t)$ of the boundary value problem* (1.1) *such that*

$$\alpha(t) \leq x(t) \leq \beta(t),\ t \in [0,1].$$

Proof. Define the following modifications of $f(t,x)$ and $h_i(x)(i = 1,2)$

$$F(t,x) = \begin{cases} f(t, \beta(t)) + \dfrac{x - \beta(t)}{1 + |x - \beta|}, & \text{if } x > \beta, \\ f(t,x), & \text{if } \alpha \leq x \leq \beta, \\ f(t, \alpha) + \dfrac{x - \alpha(t)}{1 + |x - \alpha|}, & \text{if } x < \alpha. \end{cases}$$

and

$$H_i(x) = \begin{cases} h_i(\beta(t)), & x > \beta(t), \\ h_i(x), & \alpha(t) \leq x \leq \beta(t), \\ h_i(\alpha(t)), & x < \alpha(t). \end{cases}$$

Consider the modified problem

$$\begin{aligned}
&x''(t) = F(t,x),\ t \in J = [0,1], \\
&x(0) - k_1x'(0) = \int_0^1 H_1(x(s))\,ds, \quad x(1) + k_2x'(1) = \int_0^1 H_2(x(s))\,ds.
\end{aligned} \tag{2.1}$$

Since $F(t,x) : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $H_i : \mathbb{R} \to \mathbb{R}$ are continuous and bounded, it follows that the boundary value problem (2.1) has a solution. Further, note that

$$\begin{aligned}
&\alpha''(t) \geq f(t,\alpha(t)) = F(t,\alpha(t)),\ t \in [0,1]\\
&\alpha(0) - k_1\alpha'(0) \leq \int_0^1 h_1(\alpha(s))ds = \int_0^1 H_1(\alpha(s))ds,\\
&\alpha(1) + k_2\alpha'(1) \leq \int_0^1 h_2(\alpha(s))ds = \int_0^1 H_2(\alpha(s))ds
\end{aligned}$$

and

$$\begin{aligned}
&\beta''(t) \leq f(t,\beta(t)) = F(t,\beta(t)),\ t \in [0,1]\\
&\beta(0) - k_1\beta'(0) \geq \int_0^1 h_1(\beta(s))ds = \int_0^1 H_1(\beta(s))ds,\\
&\beta(1) + k_2\beta'(1) \geq \int_0^1 h_2(\beta(s))ds = \int_0^1 H_2(\beta(s))ds
\end{aligned}$$

which imply that α and β are respectively lower and upper solutions of (2.1). Also, we note that any solution of (2.1) which lies between α and β, is a solution of (1.1). Thus, we only need to show that any solution $x(t)$ of (2.1) is such that $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in [0,1]$. Assume that $\alpha(t) \leq x(t)$ is not true on $[0,1]$. Then the function $k(t) = \alpha(t) - x(t)$ has a positive maximum at some $t = t_0 \in [0,1]$. If $t_0 \in (0,1)$, then

$$k(t_0) > 0,\ k'(t_0) = 0,\ k''(t_0) \leq 0$$

and hence

$$0 \geq k''(t_0) = \alpha''(t_0) - x''(t_0) \geq f(t_0,\alpha(t_0)) - (f(t_0,\alpha(t_0)) + \frac{x(t_0) - \alpha(t_0)}{1 + |x(t_0) - \alpha(t_0)|}) > 0,$$

a contradiction. If $t_0 = 0$, then $k(0) > 0$ and $k'(0) \leq 0$, but then the boundary conditions and the nondecreasing property of h_i gives

$$\begin{aligned}
k(0) &\leq k_1k'(0) + \int_0^1 [h_1(\alpha(s)) - H_1(x(s))]ds\\
&\leq \int_0^1 [h_1(\alpha(s)) - H_1(x(s))]ds.
\end{aligned}$$

If $x < \alpha(t)$, then $H_1(x(s)) = h_1(\alpha(s))$ and hence $k(0) \leq 0$, a contradiction. If $x > \beta(t)$, then $H_1(x(s)) = h_1(\beta(s)) \geq h_1(\alpha(s))$ which implies $k(0) \leq 0$, a contradiction. Hence $\alpha(t) \leq x(t) \leq \beta(t)$ and $H_1(x(s)) = h_1(x(s)) \geq h_1(\alpha(s))$ and again $k(0) \leq 0$, another contradiction. Similarly, if $t_0 = 1$, we get a contradiction. Thus $\alpha(t) \leq x(t)$, $t \in J$. Similarly, we can show that $x(t) \leq \beta(t)$, $t \in [0,1]$. □

Theorem 2.3. *Assume that α and β are lower and upper solutions of the boundary value problem* (1.1) *respectively. If $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $h_i : \mathbb{R} \to \mathbb{R}$ are continuous, $f_x(t,x) > 0$ for $t \in [0,1], x \in \mathbb{R}$ and $0 \leq h_i'(x) < 1$. Then $\alpha(t) \leq \beta(t)$, $t \in [0,1]$.*

Proof. Define $m(t) = \alpha(t) - \beta(t)$, $t \in [0,1]$, then $m(t) \in C^2[0,1]$ and

$$\begin{aligned} m(0) - k_1 m'(0) &\leq \int_0^1 [h_1(\alpha(s)) - h_1(\beta(s))]ds \\ m(1) + k_2 m'(1) &\leq \int_0^1 [h_2(\alpha(s)) - h_2(\beta(s))]ds. \end{aligned} \tag{2.2}$$

Assume that $m(t) \leq 0$ is not true for $t \in [0,1]$. Then $m(t)$ has a positive maximum at some $t_0 \in [0,1]$. If $t_0 \in (0,1)$, then $m(t_0) > 0$, $m'(t_0) = 0$ and $m''(t_0) \leq 0$. Using the increasing property of the function $f(t,x)$ in x, we obtain

$$f(t_0, \alpha(t_0)) \leq \alpha''(t_0) \leq \beta''(t_0) \leq f(t_0, \beta(t_0)) < f(t_0, \alpha(t_0)),$$

a contradiction. If $t_0 = 0$, then $m(0) > 0$ and $m'(0) \leq 0$. On the other hand, using the boundary conditions (2.2) and the assumption $0 \leq h_1'(x) < 1$, we have

$$\begin{aligned} m(0) \leq m(0) - k_1 m'(0) &\leq \int_0^1 [h_1(\alpha(s)) - h_1(\beta(s))]ds \leq \int_0^1 h_1'(c) m(s) ds \\ &\leq h_1'(c) \max_{t \in [0,1]} m(t) = h_1'(c) m(0) < m(0), \end{aligned} \tag{2.3}$$

a contradiction. If $t_0 = 1$, then $m(1) > 0$ and $m'(1) \geq 0$. But again, the boundary conditions (2.2) and the assumption $0 \leq h_2'(x) < 1$, gives

$$\begin{aligned} m(1) \leq m(0) + k_2 m'(1) &\leq \int_0^1 [h_2(\alpha(s)) - h_2(\beta(s))]ds \leq \int_0^1 h_2'(c) m(s) ds \\ &\leq h_2'(c) \max_{t \in [0,1]} m(t) = h_2'(c) m(1) < m(1), \end{aligned} \tag{2.4}$$

a contradiction. Hence

$$\alpha(t) \leq \beta(t), \quad t \in [0,1]$$

□

As a consequence of the theorem (2.3), we have

Corollary 2.4. *Assume that α and β are lower and upper solutions of the boundary value problem* (1.1) *respectively. If $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are continuous, $f_x(t,x) > 0$ and $0 \leq h'(x) < 1$, for $t \in [0,1]$, $x \in \mathbb{R}$. Then the solution of the boundary value problem* (1.1) *is unique.*

3. Quasilinearization Technique

Theorem 3.1. *Assume that*

- **($\mathbf{A_1}$)** *α and $\beta \in C^2[0,1]$ are respectively lower and upper solutions of* (1.1) *such that $\alpha(t) \leq \beta(t), t \in [0,1]$.*

($\mathbf{A_2}$) *$f(t,x) \in C^2[0,1] \times \mathbb{R}$ is such that $f_x(t,x) > 0$ and $f_{xx}(t,x) + \phi_{xx}(t,x) \leq 0$, where $\phi(t,x) \in C^2[0,1] \times \mathbb{R}$ and $\phi_{xx}(t,x) \leq 0$.*

($\mathbf{A_3}$) *$h_i \in C^2(\mathbb{R})$ $(i = 1,2)$ are nondecreasing, $0 \leq h_i'(x) < 1$ and $h_i''(x) \geq 0$.*

Then, there exists a monotone sequence $\{w_n\}$ of solutions converging uniformly and quadratically to the unique solution of the problem.

Proof. Define, $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ by $F(t,x) = f(t,x) + \phi(t,x)$. Then in view of (A_2), we note that $F \in C^2[0,1] \times \mathbb{R}$ and

$$F_{xx}(t,x) \leq 0. \tag{3.1}$$

For any $t \in [0,1]$, using Taylor's theorem, (3.1) and (A_3), we have

$$f(t,x) \leq F(t,y) + F_x(t,y)(x-y) - \phi(t,x) \tag{3.2}$$

and

$$h_i(x) \geq h_i(y) + h_i'(y)(x-y), \tag{3.3}$$

where $x, y \in \mathbb{R}$. Again applying Taylor's theorem to $\phi(t,x)$, we can find $\xi \in \mathbb{R}$ with $y \leq \xi \leq x$ such that

$$\phi(t,x) = \phi(t,y) + \phi_x(t,y)(x-y) + \frac{1}{2}\phi_{xx}(t,\xi)(x-y)^2, \tag{3.4}$$

which in view of (A_2) implies that

$$\phi(t,x) \leq \phi(t,y) + \phi_x(t,y)(x-y) \tag{3.5}$$

and

$$\phi(t,x) \geq \phi(t,y) + \phi_x(t,y)(x-y) - \frac{1}{2}|\phi_{xx}(t,\xi)|\|x-y\|^2, \tag{3.6}$$

where $\|x-y\| = \max_{t\in[0,1]}\{|x(t)-y(t)|\}$ denotes the supremum norm in the space of continuous functions on $[0,1]$. Using (3.6) in (3.2), we obtain

$$f(t,x) \leq f(t,y) + f_x(t,y)(x-y) + \frac{1}{2}|\phi_{xx}(t,\xi)|\|x-y\|^2. \tag{3.7}$$

Let $\Omega = \{(t,x) : t \in [0,1],\ x \in [\alpha,\beta]\}$ and define on Ω

$$g(t,x,y) = f(t,y) + f_x(t,y)(x-y) + \frac{1}{2}|\phi_{xx}(t,\xi)|\|x-y\|^2 \tag{3.8}$$

and

$$H_i(x,y) = h_i(y) + h_i'(y)(x-y). \tag{3.9}$$

Note that $g(t,x,y)$ and $H_i(x,y)$ are continuous, bounded and are such that $g_x(t,x,y) = f_x(t,y) > 0$ and $0 \le \frac{\partial}{\partial x} H_i(x,y) < 1$. Further, from {(3.7), (3.8)} and {(3.3), (3.9)}, we have the relations

$$\text{(3.10)} \qquad \begin{cases} f(t,x) \le g(t,x,y) \\ f(t,x) = g(t,x,x) \end{cases}$$

and

$$\text{(3.11)} \qquad \begin{cases} h_i(x) \ge H_i(x,y) \\ h_i(x) = H_i(x,x) \end{cases}$$

Now, set $w_0 = \alpha$ and consider the linear problem

$$\text{(3.12)} \qquad \begin{aligned} & x''(t) = g(t,x,w_0),\ t \in [0,1], \\ & x(0) - k_1 x'(0) = \int_0^1 H_1(x(s), w_0(s))ds, \\ & x(1) + k_2 x'(1) = \int_0^1 H_2(x(s), w_0(s))ds. \end{aligned}$$

Using (A_1), (3.10) and (3.11), we obtain

$$\begin{aligned} & w_0''(t) \ge f(t,w_0) = g(t,w_0,w_0),\ t \in [0,1], \\ & w_0(0) - k_1 w_0'(0) \le \int_0^1 h_1(w_0(s))ds = \int_0^1 H_1(w_0(s), w_0(s))ds, \\ & w_0(1) + k_2 w_0'(1) \le \int_0^1 h_2(w_0(s))ds = \int_0^1 H_2(w_0(s), w_0(s))ds \end{aligned}$$

and

$$\begin{aligned} & \beta''(t) \le f(t,\beta) \le g(t,\beta,w_0),\ t \in [0,1], \\ & \beta(0) - k_1 \beta'(0) \ge \int_0^1 h_1(\beta(s))ds \ge \int_0^1 H_1(\beta(s), w_0(s))ds, \\ & \beta(1) + k_2 \beta'(1) \ge \int_0^1 h_2(\beta(s))ds \ge \int_0^1 H_2(\beta(s), w_0(s))ds, \end{aligned}$$

which imply that w_0 and β are respectively lower and upper solutions of (3.12). It follows by theorems 2.2 and 2.3 that there exists a unique solution w_1 of (3.12) such that

$$w_0(t) \le w_1(t) \le \beta(t),\ t \in [0,1].$$

In view of (3.10), (3.11) and the fact that w_1 is a solution of (3.12), we note that w_1 is a lower solution of (1.1).

Now consider the problem

$$
\begin{aligned}
& x''(t) = g(t, x, w_1),\ t \in [0, 1], \\
(3.13) \qquad & x(0) - k_1 x'(0) = \int_0^1 H_1(x(s), w_1(s))ds, \\
& x(1) + k_2 x'(1) = \int_0^1 H_2(x(s), w_1(s))ds.
\end{aligned}
$$

Again we can show that w_1 and β are lower and upper solutions of (3.13) and hence by theorems (2.2, 2.3), there exists a unique solution w_2 of (3.13) such that

$$
w_1(t) \le w_2(t) \le \beta(t), \quad t \in [0, 1].
$$

Continuing this process, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$
w_0(t) \le w_1(t) \le w_2(t) \le ...w_n(t) \le \beta(t),\ t \in [0, 1]
$$

where, the element w_n of the sequence $\{w_n\}$ is a solution of the boundary value problem

$$
\begin{aligned}
x''(t) =& g(t, x, w_{n-1}),\ t \in [0, 1], \\
x(0) - k_1 x'(0) =& \int_0^1 H_1(x(s), w_{n-1}(s))ds, \\
x(1) + k_2 x'(1) =& \int_0^1 H_2(x(s), w_{n-1}(s))ds
\end{aligned}
$$

and

$$
(3.14) \qquad w_n(t) = P_n(t) + \int_0^1 G(t, s) g(s, w_n(s), w_{n-1}(s))ds,
$$

where

$$
\begin{aligned}
(3.15) \quad P_n(t) = \frac{1}{1 + k_1 + k_2}\{(1 - t + k_2) & \int_0^1 H_1(w_n(s), w_{n-1}(s))\, ds \\
& + (k_1 + t) \int_0^1 H_2(w_n(s), w_{n-1}(s))\, ds\}.
\end{aligned}
$$

Employing the standard arguments [2], it follows that the convergence of the sequence is uniform. If $x(t)$ is the limit point of the sequence, passing to the limit as $n \to \infty$, (3.14) gives

$$
x(t) = P(t) + \int_0^1 G(t, s) f(s, x(s))ds,
$$

where

$$
P(t) = \frac{1}{1 + k_1 + k_2}\{(1 - t + k_2) \int_0^1 h_1(x(s))ds + (k_1 + t) \int_0^1 h_2(x(s))ds\};
$$

that is, $x(t)$ is a solution of the boundary value problem (1.1).

Now, we show that the convergence of the sequence is quadratic. For that, set $e_n(t) = x(t) - w_n(t)$, $t \in [0,1]$. Note that, $e_n(t) \geq 0$, $t \in [0,1]$. Using Taylor's theorem and (3.9), we obtain

$$\begin{aligned} e_n(0) - k_1 e_n'(0) &= \int_0^1 [h_1(x(s)) - H_1(w_n(s), w_{n-1}(s)] ds \\ &= \int_0^1 [h_1'(w_{n-1}(s)) e_n(s) + \frac{1}{2} h_1''(\xi_1) e_{n-1}^2(s)] ds \end{aligned}$$

and

$$\begin{aligned} e_n(1) + k_2 e_n'(1) &= \int_0^1 [h_2(x(s)) - H_2(w_n(s), w_{n-1}(s))] ds \\ &= \int_0^1 [h_2'(w_{n-1}(s)) e_n(s) + \frac{1}{2} h_2''(\xi_2) e_{n-1}^2(s)] ds \end{aligned}$$

where, $w_{n-1} \leq \xi_1, \xi_2 \leq x$. In view of (A_3), there exist $\lambda_i < 1$ and $C_i \geq 0$ such that $h_i'(w_{n-1}(s)) \leq \lambda_i$ and $\frac{1}{2} h_i''(\xi_1) \leq C_i \, (i = 1, 2)$. Let $\lambda(< 1) = \max\{\lambda_1, \lambda_2\}$ and $C(\geq 0) = \max\{C_1, C_2\}$, then

$$\begin{aligned} e_n(0) - k_1 e_n'(0) &\leq \lambda \int_0^1 e_n(s) ds + C \int_0^1 e_{n-1}^2(s) ds \leq \lambda \int_0^1 e_n(s) ds + C \|e_{n-1}\|^2 \\ e_n(1) + k_2 e_n'(1) &\leq \lambda \int_0^1 e_n(s) ds + C \int_0^1 e_{n-1}^2(s) ds \leq \lambda \int_0^1 e_n(s) ds + C \|e_{n-1}\|^2. \end{aligned} \tag{3.16}$$

Further, using Taylor's theorem, (A_2), (3.5) and (3.9), we obtain

$$\begin{aligned} e_n''(t) &= x''(t) - w_n''(t) = (F(t,x) - \phi(t,x)) \\ &\quad - [f(t, w_{n-1}) + f_x(t, w_{n-1})(w_n - w_{n-1}) + \frac{1}{2} |\phi_{xx}(t,\xi)| \|w_n - w_{n-1}\|^2] \\ &\geq f_x(t, w_{n-1}) e_n(t) + F_{xx}(t, \xi_3) \frac{e_{n-1}^2(t)}{2!} - \frac{1}{2} |\phi_{xx}(t,\xi)| \|e_{n-1}\|^2 \\ &\geq -\frac{1}{2} (|F_{xx}(t,\xi_3)| + |\phi_{xx}(t,\xi)|) \|e_{n-1}\|^2 \\ &\geq -M \|e_{n-1}\|^2, \end{aligned} \tag{3.17}$$

where $w_{n-1} \leq \xi_3 \leq x$, $w_{n-1} \leq \xi \leq w_n$, $|F_{xx}| \leq M_1$, $|\phi_{xx}| \leq M_2$ and $2M = M_1 + M_2$. From (3.16) and (3.17), it follows that

$$e_n(t) \leq r(t) \text{ on } [0,1],$$

where, $r(t) \geq 0$ is the unique solution of the boundary value problem

$$r''(t) = -M\|e_{n-1}\|^2,\ t \in [0,1]$$
$$r(0) - k_1 r'(0) = \lambda \int_0^1 r(s)ds + C\|e_{n-1}\|^2$$
$$r(1) + k_2 r'(1) = \lambda \int_0^1 r(s)ds + C\|e_{n-1}\|^2.$$

Thus, $r(t) =$

$$\begin{aligned}
&= \frac{1}{1+k_1+k_2}[(1-t+k_2)(\lambda \int_0^1 r(s)ds + C\|e_{n-1}\|^2) \\
&+ (t+k_1)(\lambda \int_0^1 r(s)ds + C\|e_{n-1}\|^2)] - M \int_0^1 G(t,s)\|e_{n-1}\|^2 ds \\
&\leq \frac{1}{1+k_1+k_2}[\lambda\{(1-t+k_2)+(t+k_1)\}\|r\| + C\{(1-t+k_2)+(t+k_1)\}\|e_{n-1}\|^2) \\
&+ M\|e_{n-1}\|^2 \int_0^1 |G(t,s)|ds \\
&= \lambda\|r\| + C\|e_{n-1}\|^2 + Ml\|e_{n-1}\|^2 = \lambda\|r\| + L\|e_{n-1}\|^2,
\end{aligned}$$

where l is a bound for $\int_0^1 |G(t,s)|ds$ and $L = C + lM$. Taking the maximum over $[0,1]$, we get

$$\|r\| \leq \delta\|e_{n-1}\|^2,$$

where, $\delta = \frac{L}{1-\lambda}$. □

4. Rapid convergence

Theorem 4.1. *Assume that*

- **(B_1)** α, $\beta \in C^2[0,1]$ *are lower and upper solutions of* (1.1) *respectively such that* $\alpha(t) \leq \beta(t), t \in [0,1]$.
- **(B_2)** $f(t,x) \in C^k[[0,1] \times \mathbb{R}]$ *such that* $\frac{\partial^j}{\partial x^j} f(t,x) \geq 0\, (j = 1,2,3...,k-1)$, *and* $\frac{\partial^k}{\partial x^k}(f(t,x) + \phi(t,x)) \leq 0$, *where,* $\phi \in C^k[[0,1] \times \mathbb{R}]$ *and* $\frac{\partial^k}{\partial x^k}\phi(t,x) \leq 0$,
- **(B_3)** $h_j(x) \in C^k[\mathbb{R}]$ *such that* $\frac{d^i}{dx^i} h_j(x) \leq \frac{M}{(\beta-\alpha)^{i-1}}\ (i = 1,2,...,k-1)$ *and* $\frac{d^k}{dx^k} h_j(x) \geq 0$, *where* $M < 1/3$ *and* $j = 1,2$.

Then, there exists a monotone sequence $\{w_n\}$ of solutions converging uniformly to the unique solution of the problem. Moreover the rate of convergence is of order $k \geq 2$.

Proof. Define, $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ by $F(t,x) = f(t,x) + \phi(t,x)$, $t \in [0,1]$, then in view of (B_2), we note that $F \in C^k[[0,1] \times \mathbb{R}]$ and

$$\frac{\partial^k}{\partial x^k} F(t,x) \leq 0. \tag{4.1}$$

Using (B_3), Taylor's theorem and (4.1), we have

$$f(t,x) \le \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} F(t,y) \frac{(x-y)^i}{i!} - \phi(t,x) \tag{4.2}$$

and

$$h_j(x) \ge \sum_{i=0}^{k-1} \frac{d^i}{dx^i} h_j(y) \frac{(x-y)^i}{i!}. \tag{4.3}$$

Expanding $\phi(t,x)$ about (t,y) by Taylor's theorem, we can find $y \le \xi \le x$, such that

$$\phi(t,x) = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} \phi(t,y) \frac{(x-y)^i}{i!} + \frac{\partial^k}{\partial x^k} \phi(t,\xi) \frac{(x-y)^k}{k!}, \tag{4.4}$$

which in view of (B_2) implies that

$$\phi(t,x) \le \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} \phi(t,y) \frac{(x-y)^i}{i!}. \tag{4.5}$$

Using (4.4) in (4.2), we obtain

$$f(t,x) \le \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t,y) \frac{(x-y)^i}{i!} - \frac{\partial^k}{\partial x^k} \phi(t,\xi) \frac{(x-y)^k}{k!}. \tag{4.6}$$

Let $\Omega = \{(t,x) : t \in [0,1],\, x \in [\alpha,\beta]\}$ and define on Ω the functions

$$g^*(t,x,y) = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t,y) \frac{(x-y)^i}{i!} - \frac{\partial^k}{\partial x^k} \phi(t,\xi) \frac{(x-y)^k}{k!} \tag{4.7}$$

and

$$H_j^*(x,y) = \sum_{i=0}^{k-1} \frac{d^i}{dx^i} h_j(y) \frac{(x-y)^i}{i!}. \tag{4.8}$$

Then, we note that $g^*(t,x,y)$ and $H_j^*(x,y)$ are continuous, bounded and are such that

$$g_x^*(t,x,y) = \sum_{i=1}^{k-1} \frac{\partial^i}{\partial x^i} f(t,y) \frac{(x-y)^{i-1}}{(i-1)!} - \frac{\partial^k}{\partial x^k} \phi(t,\xi) \frac{(x-y)^{k-1}}{(k-1)!} \ge 0$$

and

$$\begin{aligned} \frac{\partial}{\partial x} H_j^*(x,y) &= \sum_{i=1}^{k-1} \frac{d^i}{dx^i} h_j(y) \frac{(x-y)^{i-1}}{(i-1)!} \\ &\le \sum_{i=1}^{k-1} \frac{M}{(\beta-\alpha)^{i-1}} \frac{(\beta-\alpha)^{i-1}}{(i-1)!} \le M(3 - \frac{1}{2^{k-2}}) < 1. \end{aligned}$$

Further, from {(4.6),(4.7) } and {(4.3),(4.8) }, we have the relations

$$\begin{cases} f(t,x) \le g^*(t,x,y), \\ f(t,x) = g^*(t,x,x) \end{cases} \tag{4.9}$$

and

$$
(4.10) \qquad \begin{cases} h_j(x) \geq H_j^*(x,y), \\ h_j(x) = H_j^*(x,x). \end{cases}
$$

Now, set $\alpha = w_0$ and consider the linear problem

$$
(4.11) \qquad \begin{aligned} x''(t) &= g^*(t,x,w_0),\ t \in [0,1], \\ x(0) - k_1 x'(0) &= \int_0^1 H_1^*(x(s), w_0(s))ds, \\ x(1) + k_2 x'(1) &= \int_0^1 H_2^*(x(s), w_0(s))ds. \end{aligned}
$$

The assumption (B_1) and the expressions (4.9), (4.10) yields

$$
\begin{aligned} & w_0''(t) \geq f(t,w_0) = g^*(t,w_0,w_0),\ t \in [0,1], \\ & w_0(0) - k_1 w_0'(0) \leq \int_0^1 h_1(w_0(s))ds = \int_0^1 H_1^*(w_0(s), w_0(s))ds, \\ & w_0(1) + k_2 w_0'(1) \leq \int_0^1 H_2^*(w_0(s))ds = \int_0^1 H_2^*(w_0(s), w_0(s))ds \end{aligned}
$$

and

$$
\begin{aligned} & \beta''(t) \leq f(t,\beta) \leq g(t,\beta,w_0),\ t \in [0,1], \\ & \beta(0) - k_1 \beta'(0) \geq \int_0^1 h_1(\beta(s))ds \geq \int_0^1 H_1^*(\beta(s), w_0(s))ds, \\ & \beta(1) + k_2 \beta'(1) \geq \int_0^1 h_2(\beta(s))ds \geq \int_0^1 H_2^*(\beta(s), w_0(s))ds, \end{aligned}
$$

imply that w_0 and β are respectively lower and upper solutions of (4.11). Hence by theorems (2.2, 2.3), there exists a unique solution w_1 of (4.11) such that

$$
w_0(t) \leq w_1(t) \leq \beta(t),\ t \in [0,1].
$$

Continuing this process, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$
w_0(t) \leq w_1(t) \leq w_2(t) \leq ...w_n(t) \leq \beta(t),\ t \in [0,1],
$$

where the element w_n of the sequence $\{w_n\}$ is a solution of the boundary value problem

$$
\begin{aligned} x''(t) =& g^*(t,x,w_{n-1}),\ t \in [0,1], \\ x(0) - k_1 x'(0) =& \int_0^1 H_1^*(x(s), w_{n-1}(s))ds, \\ x(1) + k_2 x'(1) =& \int_0^1 H_2^*(x(s), w_{n-1}(s))ds. \end{aligned}
$$

By the same process as in theorem (3.1), we can show that the sequence converges uniformly to the unique solution of (1.1).
Now, we show that the convergence of the sequence is of order $k \geq 2$. For that, set

$$e_n(t) = x(t) - w_n(t) \text{ and } a_n(t) = w_{n+1}(t) - w_n(t),\ t \in [0,1].$$

Then,

$$e_n(t) \geq 0,\ a_n \geq 0,\ e_{n+1} = e_n - a_n,\ e_n^i \geq a_n^i\ (i = 1, 2...)$$

and

$$\begin{aligned} e_n(0) - k_1 e_n'(0) &= \int_0^1 [h_1(x(s)) - H_1^*(w_n(s), w_{n-1}(s)]ds \\ e_n(1) + k_2 e_n'(1) &= \int_0^1 [h_2(x(s)) - H_2^*(w_n(s), w_{n-1}(s)]ds. \end{aligned} \tag{4.12}$$

Using Taylor's theorem and (4.8), we obtain

$$\begin{aligned} h_j(x(s)) - H_j^*(w_n(s), w_{n-1}(s) &= \sum_{i=0}^{k-1} \frac{d^i}{dx^i} h_j(w_{n-1}) \frac{(x - w_{n-1})^i}{i!} + \frac{d^k}{dx^k} h_j(c) \frac{(x - w_{n-1})^k}{k!} \\ &- \sum_{i=0}^{k-1} \frac{d^i}{dx^i} h_j(w_{n-1}) \frac{(w_n - w_{n-1})^i}{i!} \\ &= (\sum_{i=1}^{k-1} \frac{d^i}{dx^i} h_j(w_{n-1}) \frac{1}{i!} \sum_{l=0}^{i-1} e_{n-1}^{i-1-l} a_{n-1}^l) e_n + \frac{d^k}{dx^k} h_j(c) \frac{e_{n-1}^k}{k!} \\ &\leq p_j(t) e_n(t) + \frac{M}{\gamma^{k-1}} \frac{e_{n-1}^k}{k!} \leq p_j(t) e_n(t) + \frac{M}{\gamma^{k-1}} \frac{\|e_{n-1}^k\|}{k!}, \end{aligned}$$

where $p_j(t) = \sum_{i=1}^{k-1} \frac{d^i}{dx^i} h_j(w_{n-1}) \frac{1}{i!} \sum_{l=0}^{i-1} e_{n-1}^{i-1-l} a_{n-1}^l$ and $\gamma = \max_{t \in [0,1]} \beta(t) - \min_{t \in [0,1]} \alpha(t)$. In view of (B_3), we have

$$p_j(t) \leq \sum_{i=1}^{k-1} \frac{M}{(\beta - \alpha)^{i-1}} \frac{1}{i!} \sum_{l=0}^{i-1} e_{n-1}^{i-1} \leq \sum_{i=1}^{k-1} \frac{M}{(\beta - \alpha)^{i-1}} \frac{1}{(i-1)!} (\beta - \alpha)^{i-1} < 1.$$

It follows that, we can find $\lambda < 1$ such that $p_j(t) \leq \lambda$, $t \in [0,1]$, $(j = 1, 2)$ and hence

$$\begin{aligned} e_n(0) - k_1 e_n'(0) &\leq \lambda \int_0^1 e_n(s) ds + \frac{M}{\gamma^{k-1} k!} \|e_{n-1}\|^k \\ e_n(1) + k_2 e_n'(1) &\leq \lambda \int_0^1 e_n(s) ds + \frac{M}{\gamma^{k-1} k!} \|e_{n-1}\|^k. \end{aligned} \tag{4.13}$$

Now, using Taylor's theorem and (4.5), we obtain

$$
\begin{aligned}
e_n''(t) &= x''(t) - w_n''(t) \\
&= [F(t,x) - \phi(t,x)] - [\sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_{n-1}) \frac{a_{n-1}^i}{i!} - \frac{\partial^k}{\partial x^k}\phi(t,\xi)\frac{a_{n-1}^k}{k!}] \\
&= \sum_{i=1}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_{n-1}) \frac{(e_{n-1}^i - a_{n-1}^i)}{i!} + \frac{\partial^k}{\partial x^k} F(t, c_1) \frac{e_{n-1}^k}{k!} + \frac{\partial^k}{\partial x^k}\phi(t,\xi))\frac{a_{n-1}^k}{k!} \qquad (4.14) \\
&\geq \sum_{i=1}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_{n-1}) \frac{\sum_{l=0}^{i-1} e_{n-1}^{i-1-l} a_{n-1}^l}{i!} e_n + \frac{e_{n-1}^k}{k!}(\frac{\partial^k}{\partial x^k} F(t, c_1) + \frac{\partial^k}{\partial x^k}\phi(t,\xi)) \\
&\geq -N\frac{\|e_{n-1}\|^k}{k!},
\end{aligned}
$$

where $-N_1 \leq \frac{\partial^k}{\partial x^k} F(t,x) \leq 0$, $-N_2 \leq \frac{\partial^k}{\partial x^k}\phi(t,x) \leq 0$ and $N = \max\{N_1, N_2\}$. From (4.13) and (4.14), it follows that $0 \leq e_n(t) \leq r(t), t \in [0,1]$, where $r(t)$ is the unique solution of the problem

$$
\begin{aligned}
r''(t) &= -N\frac{e_{n-1}^k}{k!}, t \in [0,1] \\
r(0) - k_1 r'(0) &= \lambda \int_0^1 r(s)ds + \frac{M}{\gamma^{k-1}k!}\|e_{n-1}\|^k \\
r(1) + k_2 r'(1) &= \lambda \int_0^1 r(s)ds + \frac{M}{\gamma^{k-1}k!}\|e_{n-1}\|^k
\end{aligned}
$$

and

$$
\begin{aligned}
r(t) = {} & \frac{1}{1+k_1+k_2}[(1-t+k_2)(\lambda\int_0^1 r(s)\,ds + \frac{M}{\gamma^{k-1}k!}\|e_{n-1}\|^k) + (t+k_1)(\lambda\int_0^1 r(s)\,ds \\
& + \frac{M}{\gamma^{k-1}k!}\|e_{n-1}\|^k] - N\int_0^1 G(t,s)\frac{\|e_{n-1}\|^k}{k!}\,ds \\
& \leq \frac{1}{1+k_1+k_2}[\lambda\{(1-t+k_2) + (t+k_1)\}\|r\| \\
& + \{(1-t+k_2) + (t+k_1)\}\frac{M}{\gamma^{k-1}k!}\|e_{n-1}\|^k] + N\|e_{n-1}\|^2 \int_0^1 |G(t,s)|\,ds \\
& = \lambda\|r\| + C'\|e_{n-1}\|^k,
\end{aligned}
$$

where L is a bound for $\int_0^1 |G(t,s)|ds$ and $C' = \frac{M}{\gamma^{k-1}k!} + NL$. Taking the maximum over $[0,1]$, we get

$$\|r\| \leq \delta\|e_{n-1}\|^k,$$

where, $\delta = \frac{C'}{1-\lambda}$. □

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